

An Exact Cosmological Solution to String Theory

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Abstract

A homogeneous anisotropic four dimensional spacetime with Lorentzian signature is constructed from an ungauged WZW model based on a non-semisimple Lie group. The associated non-linear σ -model describes string propagation in an expanding-contracting universe with antisymmetric tensor and dilaton backgrounds. The current algebra of $SL(2, \mathbb{R}) \times \mathbb{R}$ is constructed in terms of two free boson fields and two generalized parafermions, or four free bosons with background charge. This representation is used to study the string spectrum in the cosmological background.

1 Introduction

String propagation on a four dimensional plane wave background was recently described with a WZW model constructed on the centrally extended Euclidean group in two dimensions [1]. The corresponding current algebra was considered by Kiritsis and Kounnas [2], who used the representations to construct the spectrum and scattering of strings moving in gravitational wave backgrounds.

In this article, we present an example of a homogeneous, anisotropic, four-dimensional space-time, with Lorentz signature, that can be constructed from an ungauged WZW model based on a non-semisimple Lie group. This space-time, with the corresponding antisymmetric tensor and dilaton fields, can be considered as an exact cosmological solution to string theory. The associated σ -model describes string propagation on an expanding and contracting space-time which begins from a collapsed state, (zero volume), and recollapses after a period of time proportional to the level k of the affine Kac-Moody algebra. This geometry is not asymptotically flat, and thus a well defined scattering matrix cannot be constructed.

We study the current algebra corresponding to $SL(2, R) \times R$, which can be reduced to the central extension of the 2-d Euclidean group E_2 , through an unconventional contraction. The conformal field theory (CFT) description of the model reveals several differences from current algebras previously considered in the literature [3]-[7]. A systematic description of current algebras based on non semisimple groups was performed in [8].

The algebra may be described in terms of two free bosonic fields without background charge, plus two generalized parafermionic fields [16], or equivalently in terms of four free bosons with background charge. We construct one representation which may be used to study the spectrum and scattering amplitudes of bosonic strings moving in the cosmological background.

The central charge of the CFT turns out to be non-integer (in general), and depends on the level of the affine algebra. This differs from what happens to other similar constructions considered so far, in which the central charge is integer and equals the dimension of the group manifold [3]-[7]. The reason being that the bilinear form entering the operator product expansion of the current algebra is different from the metric on the Lie algebra, i.e., the bilinear form which raises and lowers group indices [8].

The paper is organized as follows. In Section 2 we construct the WZW model on the group $SL(2, R) \times R$ and identify the background fields from the associated σ model. In Section 3 we examine some issues of duality in the σ model picture. The Sugawara construction on the nonsemisimple group is performed in Section 4 where the central charge is computed. Finally, in Section 5 we analyze further the structure of the current algebra mapping the currents into bosonic and parafermionic fields. We also construct irreducible representations of the $SL(2, R) \times R$ Lie algebra.

2 The WZW model on $SL(2, R) \times R$

Let us consider the WZW model constructed on a certain non-semisimple Lie algebra of dimension four. The $SL(2, R)$ generators J, P_1, P_2 , satisfy the algebra,

$$[P_a, P_b] = -\Lambda \epsilon_{ab} J \quad [J, P_a] = \epsilon_{ab} P_b$$

for $\Lambda \neq 0$, and due to the well-known ambiguity of the two-dimensional angular momentum [9], J may be replaced by $J - sT$, and Λ by Λ/s , with T a central extension. When s is set to infinity, the central extension of the two-dimensional Euclidean group E_2^c is obtained [1]. However, instead of taking that contraction, we redefine P_a as $\sqrt{\Lambda} P_a$, and use this algebra, $(SL(2, R) \times R)$, to construct the WZW model. Namely,

$$[P_a, P_b] = -\epsilon_{ab} J \quad [J, P_a] = \epsilon_{ab} P_b \quad [T, J] = [T, P_a] = 0 \quad (1)$$

In general, given a Lie algebra with generators T_a (here $T_a = P_1, P_2, J, T$) and structure constants f_{ab}^c , to define a WZW model one needs a bilinear form Ω_{ab} in the generators T_a , which is symmetric, invariant

$$f_{ab}^d \Omega_{cd} + f_{ac}^d \Omega_{bd} = 0 \quad (2)$$

and non-degenerate, so that there exists an inverse matrix Ω^{ab} , to raise and lower group indices. Therefore,

$$S_{WZW}(g) = \frac{1}{4\pi} \int_{\Sigma} d^2\sigma \Omega_{ab} A_{\alpha}^a A^{b\alpha} + \frac{i}{12\pi} \int_B d^3\sigma \epsilon_{\alpha\beta\gamma} A^{a\alpha} A^{b\beta} A^{c\gamma} \Omega_{cd} f_{ab}^d \quad (3)$$

where the fields A_{α}^a are defined through $g^{-1} \partial_{\alpha} g = A_{\alpha}^a T_a$. Here B is a three-manifold with boundary $\partial B = \Sigma$, and g is a map of Σ to the Lie group,

extended to a map from B . In order to construct the WZW action, a necessary condition is the existence of the invariant metric Ω_{ab} . Usually for semisimple groups one can take the Cartan-Killing form $\tilde{\Omega}_{ab} = f_{ac}^d f_{bd}^c$, which is equivalent to $Tr T_a T_b$, with the trace taken in the adjoint representation [10]. However, for non-semisimple groups this quadratic form is degenerate,

$$\tilde{\Omega}_{ab} = 2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4)$$

(the only non-zero structure constants are $f_{23}^1 = -f_{13}^2 = -f_{12}^3 = 1$). Nevertheless, the Lie algebra (1) has a non-degenerate invariant metric, namely, the most general solution of equation (2),

$$\Omega_{ab} = k \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \quad (5)$$

where k and λ are free parameters. The metric on the Lie algebra has Lorentzian signature if $\lambda > 0$, and that will be the signature of the space-time described by the corresponding σ model.

In order to write the WZW action we need to construct the group elements by exponentiating the algebra and parametrizing the group manifold with coordinates t, x, y and z . I.e., by writing the elements of the group as

$$g = e^{xP_1} e^{tJ} e^{yP_1 + zT} \quad (6)$$

and using the relations

$$\begin{aligned} e^{-tJ} P_1 e^{tJ} &= \cos t \cdot P_1 - \sin t \cdot P_2 \\ e^{-yP_1} P_2 e^{yP_1} &= \cosh y \cdot P_2 + \sinh y \cdot J \\ e^{-yP_1} J e^{yP_1} &= \cosh y \cdot J + \sinh y \cdot P_2 \end{aligned}$$

together with

$$\partial_\alpha e^H = \int_0^1 dx e^{xH} \partial_\alpha H e^{(1-x)H}$$

We can then compute

$$\begin{aligned}
g^{-1}\partial_\alpha g &= (\partial_\alpha y + \cos t \partial_\alpha x)P_1 + (\sinh y \partial_\alpha t - \sin t \cosh y \partial_\alpha x)P_2 + \\
&+ (\cosh y \partial_\alpha t - \sin t \sinh y \partial_\alpha x)J + \partial_\alpha z T
\end{aligned} \tag{7}$$

from which we can read off the elements of the algebra $g^{-1}\partial_\alpha g = A_\alpha^a T_a$,

$$\begin{aligned}
A_\alpha^1 &= \partial_\alpha y + \cos t \partial_\alpha x \\
A_\alpha^2 &= \sinh y \partial_\alpha t - \sin t \cosh y \partial_\alpha x \\
A_\alpha^3 &= \cosh y \partial_\alpha t - \sin t \sinh y \partial_\alpha x \\
A_\alpha^4 &= \partial_\alpha z
\end{aligned}$$

Thus, the terms that are integrated in the action (3) may be computed to be

$$\begin{aligned}
\Omega_{ab} A_\alpha^a A^{b\alpha} &= k(A_\alpha^1 A^{1\alpha} + A_\alpha^2 A^{2\alpha} - A_\alpha^3 A^{3\alpha} + \lambda A_\alpha^4 A^{4\alpha}) \\
&= k(-\partial_\alpha t \partial^\alpha t + \partial_\alpha x \partial^\alpha x + 2 \cos t \partial_\alpha x \partial^\alpha y + \partial_\alpha y \partial^\alpha y + \lambda \partial_\alpha z \partial^\alpha z)
\end{aligned}$$

and

$$\begin{aligned}
\epsilon_{\alpha\beta\gamma} A^{a\alpha} A^{b\beta} A^{c\gamma} \Omega_{cd} f_{ab}^d &= 3k \epsilon_{\alpha\beta\gamma} A^{1\alpha} (A^{2\beta} A^{3\gamma} - A^{3\beta} A^{2\gamma}) \\
&= 6k \epsilon^{\alpha\beta\gamma} \sin t \partial_\alpha y \partial_\beta t \partial_\gamma x
\end{aligned}$$

which can be written in the form

$$= 6k \epsilon^{\alpha\beta\gamma} \partial_\alpha (\cos t \partial_\beta y \partial_\gamma x)$$

Therefore, the Wess-Zumino term may be reduced to an integral over Σ , without introducing singularities,

$$\Gamma = 6k \int_B d^3 \sigma \epsilon^{\alpha\beta\gamma} \partial_\alpha (\cos t \partial_\beta y \partial_\gamma x) = 6k \int_\Sigma d^2 \sigma \epsilon^{\beta\gamma} \cos t \partial_\beta y \partial_\gamma x$$

Finally the action looks like

$$\begin{aligned}
S_{WZW} &= \frac{k}{4\pi} \int_\Sigma d^2 \sigma [-\partial_\alpha t \partial^\alpha t + \partial_\alpha x \partial^\alpha x + \partial_\alpha y \partial^\alpha y + 2 \cos t \partial_\alpha x \partial^\alpha y + \lambda \partial_\alpha z \partial^\alpha z + \\
&+ 2i \epsilon^{\beta\gamma} \cos t \partial_\beta y \partial_\gamma x]
\end{aligned} \tag{8}$$

One may read off the space-time metric, antisymmetric tensor and dilaton fields by identifying the WZW action with the σ model action

$$S = \int d^2\sigma [G_{\mu\nu} \partial_\alpha X^\mu \partial^\alpha X^\nu + i B_{\mu\nu} \epsilon_{\alpha\beta} \partial^\alpha X^\mu \partial^\beta X^\nu + \Phi^{(2)} R]$$

where $X^\mu = (t, x, y, z)$. The space-time geometry is described by a Lorentz signature metric, (for $\lambda > 0$), which is homogeneous but anisotropic,

$$G_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & \cos t & 0 \\ 0 & \cos t & 1 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \quad (9)$$

The only non-zero component of the antisymmetric field is $B_{xy} = \cos t$, and there is also a constant dilaton background field due to the homogeneity of the group manifold. This metric defines a cosmological model of type I, according to the Bianchi classification, with t playing the role of time parameter, running in the range $0 < t < \pi$. At $t = 0$, the universe begins in a collapsed state, since the determinant of the metric vanishes. At $t = \pi$, it recollapses again, because of the same reason. A factor $k/4\pi$ has been suppressed from the metric (9), and so the actual time scale for the expansion and recontraction is proportional to k . The maximum spatial volume of this universe is reached at $t = \pi/2$ and is again of order k and proportional to λ , (which may be thought of as a scale factor in the z direction). The non vanishing components of the Riemann tensor are $R_{1212} = R_{1313} = -\frac{1}{4}$ and $R_{1213} = R_{2323} = \left(\frac{\sin t}{2}\right)^2$. The Ricci tensor is $R_{\mu\nu} = -\frac{1}{2} (G_{\mu\nu} - \lambda \delta_\mu^z \delta_\nu^z)$ and the scalar curvature $R = -\frac{3}{2}$ is constant.

This model, being a WZW model, is conformally invariant, and thus the background satisfies the β -function equations of the non-linear σ -model to all orders in the α' expansion, (here $\alpha' = 1/k$). Thus, the solution differs from the homogeneous Bianchi geometries recently found in references [13] as solutions of the first order β -functions. The central charge receives quantum corrections, therefore, unlike the models considered so far [3]-[7], it does not equal the dimension of the group manifold and depends on the level k .

Exact metric and dilaton backgrounds were found in reference [11], using the conformal invariance of the $SL(2, R) \times SO(1, 1)^{(d-2)}/SO(1, 1)$ coset models. However, the method introduced by Sfetsos is incapable of determining the exact antisymmetric tensor.

Consistent string propagation requires unitarity at the quantum level in addition to conformal invariance of the corresponding non linear σ model [12]. However, before examining the conditions for decoupling of zero norm states, we analyze below some issues of duality in the σ model picture.

3 Some Duality Considerations.

The action (8) has several Killing symmetries, i.e., there are three explicit isometries realized by translations $x \rightarrow x + a$, $y \rightarrow y + b$, $z \rightarrow z + c$, with a, b, c constants. So, there is in principle an $O(3, 3)$ duality symmetry. Let us analyze a duality transformation in an arbitrary direction, namely in the plane defined by x and y . We first make a rotation $(x, y) \rightarrow (x', y')$, as

$$x' = \cos \rho \cdot x - \sin \rho \cdot y$$

$$y' = \sin \rho \cdot x + \cos \rho \cdot y$$

with ρ an arbitrary angle in the range $-\pi/2 \leq \rho \leq \pi/2$. Now we can make the duality transformation in the x' direction, characterized by ρ . The dual metric, antisymmetric tensor and dilaton fields will depend on this free parameter ρ , and they are given by

$$\begin{aligned} \tilde{G}_{xx} &= \frac{1}{G_{xx}} = \frac{1}{1 + \sin 2\rho \cos t} \\ \tilde{G}_{xy} &= \frac{B_{xy}}{G_{xx}} = \frac{\cos t}{1 + \sin 2\rho \cos t} \end{aligned} \tag{10}$$

$$\begin{aligned} \tilde{G}_{yy} &= G_{yy} - \frac{G_{xy}^2 - B_{xy}^2}{G_{xx}} = \frac{1}{1 + \sin 2\rho \cos t} \\ \tilde{B}_{xy} &= \frac{G_{xy}}{G_{xx}} = \frac{\cos 2\rho \cos t}{1 + \sin 2\rho \cos t} \end{aligned} \tag{11}$$

$$\tilde{\Phi} = \Phi - \ln(G_{xx}) = \Phi - \ln[1 + \sin(2\rho) \cos t] \tag{12}$$

where G_{ij} , B_{ij} , and Φ are expressed in the coordinates x' and y' . The determinant and the scalar curvature of the dual metric are

$$\det \tilde{G} = \frac{-\lambda(\sin t)^2}{(1 + \sin 2\rho \cos t)^2}$$

$$\tilde{R} = \frac{1 - 7 \cos(4\rho) + 8 \sin(2\rho) \cos t}{4 (1 + \sin(2\rho) \cos t)^2} \quad (13)$$

and they show that the spacetime begins at $t = 0$ from a collapsed state, (zero volume). When the duality is performed in the direction determined by $\rho = \pi/4$, there is no initial curvature singularity since $\tilde{R} = (\sec(\frac{t}{2}))^2$. When $t \rightarrow \pi$, $\det \tilde{G}$ and $\tilde{R} \rightarrow \infty$. For $\rho = -\pi/4$ the determinant diverges when $t \rightarrow 0$ and vanishes for $t \rightarrow \pi$, (the spacetime recollapses), while the curvature \tilde{R} diverges for $t = 0$ but it is finite when $t = \pi$. Recall that the original spacetime has no curvature singularities.

From eqs. (10), (11) and (12), the background may be seen to be self-dual for the particular values $\rho = 0, \pm\pi/2$, i.e., when the duality is performed in the original x or y directions. Obviously, the same behaviour occurs if the duality is performed in the z direction, for $\lambda = 1$.

4 Current Algebra of the Conformal Model

Current algebra is a useful tool to understand conformal field theories and string theory [14]. The WZW models are simple because they realize current algebra as its full symmetry. The action (3) is invariant under an infinitesimal transformation of the form

$$g \longrightarrow g + \bar{\epsilon}g + g\epsilon$$

where $\epsilon(z) = \epsilon^a(z)T_a$ and $\bar{\epsilon}(\bar{z}) = \bar{\epsilon}^a(\bar{z})T_a$, in complex coordinates (z, \bar{z}) . The Noether currents associated to this symmetry, and to some Lie algebra element, are

$$J_z^a = \frac{1}{4\pi} \Omega^{ab} A_{zb} \quad , \quad J_{\bar{z}}^a = \frac{1}{4\pi} \Omega^{bc} A_{\bar{z}b} V_c^a$$

so that $J(z) = J_z^a T_a$ and $\bar{J}(\bar{z}) = J_{\bar{z}}^a T_a$. V_c^a is defined through

$$V_b^a T^b = g^{-1} T^a g$$

J_z^a and $J_{\bar{z}}^a$ are holomorphic and antiholomorphic currents, respectively. These bosonic currents satisfy two copies of the current algebra given by the following operator product expansion (OPE) [15],

$$J_a(z) J_b(w) = \frac{\Omega_{ab}}{(z-w)^2} + f_{ab}^c \frac{J_c(w)}{(z-w)} + regular \quad (14)$$

where $J_a = (P_1, P_2, J, T)$. The bilinear form Ω_{ab} must be symmetric and invariant and the Jacobi identity states that $f_{abc} = f_{ab}^d \Omega_{cd}$ is completely antisymmetric. In our case Ω_{ab} is given by (5). (In the semisimple cases $\tilde{\Omega}_{ab}$ would be used instead of Ω_{ab}).

Once we have the current algebra, we can construct the stress tensor that is bilinear in the currents,

$$T(z) = L^{ab} : J_a J_b : (z)$$

with L^{ab} a symmetric matrix determined by requiring that $T(z)$ realizes the Virasoro algebra, i.e.,

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + regular \quad (15)$$

and that the currents $J^a(z)$ are primary fields of conformal weight 1 with respect to the stress tensor $T(z)$, i.e.,

$$T(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)} + regular \quad (16)$$

Equation (16) implies that the current symmetry remains unchanged in the quantum theory. Therefore, we get the following equations for the matrix L^{ab} ,

$$\begin{aligned} L^{cb} f_{ba}^e + L^{eb} f_{ba}^c &= 0 \\ 2L^{cb} \Omega_{ba} + L^{bd} f_{ab}^e f_{ed}^c &= \delta_a^c \end{aligned}$$

The first equation is equivalent to (2). Thus, L^{ab} has the same form as Ω^{ab} . The second equation leads, uniquely, to

$$L_{ab} = 2(k+1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{k\lambda}{k+1} \end{pmatrix} \quad (17)$$

and L^{ab} is the inverse of L_{ab} ,

$$L^{ab} = \frac{1}{2} \Omega^{ab} - \delta L^{ab} \quad with \quad \delta L^{ab} = \frac{1}{2(k+1)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (18)$$

It may be seen from the last equation (18), that the stress tensor receives quantum corrections to its classical value $\frac{1}{2}\Omega^{ab}$. Then the central charge c will receive quantum corrections as well, namely

$$c(L) = 2\Omega_{ab}L^{ab} = 2\Omega_{ab}\left(\frac{1}{2}\Omega^{ab} - \delta L^{ab}\right) = 4 - 2\Omega_{ab}\delta L^{ab}$$

$$c = 4 - \frac{3}{k+1}$$

Recall that for non compact groups, in particular for the present $SL(2, R)$ case, the replacement $k \rightarrow -k$ has to be performed [17]; then

$$c = 4 + \frac{3}{k-1} \quad (19)$$

Therefore, the stress tensor may be written as

$$T(z) = \frac{-1}{2(k-1)} \left[: P_1^2 : + : P_2^2 : - : J^2 : \right] - \frac{1}{2k\lambda} : T^2 : \quad (20)$$

5 Representations of the Current Algebra

We will proceed to analyze the structure of $SL(2, R) \times R$ further, by bosonizing the Cartan subalgebra generated by J and T . We can define J^+ and J^- as a linear combination of J and T ,

$$J^+ = C_1 J + C_2 T \quad (21)$$

$$J^- = C_3 J + C_4 T$$

where the C_i are coefficients to be determined by requiring that J^+ and J^- diagonalize the Cartan subalgebra, i.e.,

$$J^+(z) J^+(w) = \frac{-1}{(z-w)^2} + regular$$

$$J^-(z) J^-(w) = \frac{\mu}{(z-w)^2} + regular \quad (22)$$

$$J^+(z) J^-(w) = regular$$

where $\mu = +1$ for Lorentzian and -1 for Euclidean signature, as we shall see below. From the OPEs above we get three equations relating the coefficients C_i ,

$$\begin{aligned} C_1^2 - \lambda C_2^2 &= -\frac{1}{k} \\ C_3^2 - \lambda C_4^2 &= \frac{\mu}{k} \\ C_1 C_3 - \lambda C_2 C_4 &= 0 \end{aligned} \tag{23}$$

Thus, we may express J^+ and J^- in terms of two bosonic fields x^0 and x^3 , as

$$J^+(z) = \partial x^3, \quad J^-(z) = \partial x^0 \tag{24}$$

In order to reproduce the OPE's (22), the two bosons must have propagators

$$\langle x^3(z) x^3(w) \rangle = -\mu \langle x^0(z) x^0(w) \rangle = -\ln(z-w) \quad , \quad \langle x^0(z) x^3(w) \rangle = 0$$

We define, in addition, two operators that will act as raising and lowering generators

$$P^\pm = P_1 \pm iP_2$$

Using the current algebra given by the OPE (14), we may calculate

$$J^+(z) P^\pm(w) = \mp i C_1 \frac{P^\pm(w)}{(z-w)} + regular \tag{25}$$

$$J^-(z) P^\pm(w) = \mp i C_3 \frac{P^\pm(w)}{(z-w)} + regular \tag{26}$$

so that P^\pm are charged under the Cartan subalgebra. Similarly, we compute

$$P^+(z) P^-(w) = \frac{-2k}{(z-w)^2} + \frac{2iJ(w)}{(z-w)} + regular \tag{27}$$

$$P^\pm(z) P^\pm(w) = regular \tag{28}$$

Now, $P^\pm(z)$ can be represented in terms of x^0 and x^3 as

$$P^\pm(z) =: e^{\pm i(C_1 x^3 - C_3 x^0)} : V^\pm(z) \tag{29}$$

Then, the OPE's (25) and (26) imply that V^\pm do not depend on x^0 and x^3 . Defining $X^- = C_1 x^3 - C_3 x^0$, we find

$$P^+(z) P^-(w) = V^+(z) V^-(w) \cdot (z-w)^{-C_1^2 + \mu C_3^2} \times \\ \times \left[1 + i \partial_w X^-(z-w) + \frac{i}{2} \partial_w^2 X^-(z-w)^2 - \frac{1}{2} (\partial_w X^-)^2 (z-w)^2 + \dots \right]$$

and using equations (23) we get $C_1^2 - \mu C_3^2 = -1/k$. This result and the OPE's (27) and (28) imply that $V^+ V^-$ must be of the following form

$$V^+(z) V^-(w) = (z-w)^{-\frac{1}{k}} \left[\frac{-2k}{(z-w)^2} - 2k A T(w) + O(z-w) \right] \quad (30)$$

where A is a coefficient and $T(w)$ does not depend on x^0 and x^3 , but will contribute to the stress tensor, as we shall show below.

There is a representation of the algebra (30) in terms of the generalized parafermions ψ_K introduced by Lykken [16], (see also [17]-[20]). These parafermions form an infinite family of fields for non compact groups, which satisfy the following OPE

$$\psi_l(z) \psi_l^\dagger(w) \sim (z-w)^{-2\Delta_l} \left[1 + \frac{2\Delta_l}{c_p} (z-w)^2 T_p(w) + O(z-w)^3 \right] \quad (31)$$

where $\psi_l^\dagger = \psi_{K-l}$. The operator $T_p(z)$ is the stress tensor of the parafermionic model with central charge $c_p = 2(K+1)/(K-2)$. Then we have,

$$T_p(z) \psi_l(w) = \frac{\Delta_l \psi_l(w)}{(z-w)^2} + \frac{\partial_w \psi_l(w)}{(z-w)} + O(1)$$

where Δ_l is the conformal dimension given by

$$\Delta_l = \frac{l(K+l)}{K} \quad (32)$$

Comparing expressions (30) and (31), we find $K = 2k$, $l = 1$ and $T(w) = T_p(w)$. Then, V^\pm can be represented in terms of ψ_1 and ψ_1^\dagger , which, from now on, we denote as $\psi_{\pm 1}$,

$$V^\pm(z) = i\sqrt{2k} \psi_{\pm 1}(z)$$

Finally, we are able to represent the P^\pm currents in terms of two bosons and two parafermions,

$$P^\pm = i\sqrt{2k}e^{\pm iX^-}\psi_{\pm 1}$$

Let us express the Sugawara stress tensor (20) in terms of these fields. A straightforward computation shows that

$$: P_1^2 + P_2^2 := \frac{1}{2} (P^+ P^- + P^- P^+) = -4k \frac{\Delta_1}{c_p} T_p + k (\partial X^-)^2 \quad (33)$$

and $\Delta_1/c_p = (k-1)/2k$. The next step is to express J^2 and T^2 in terms of ∂x^0 and ∂x^3 using equations (21) and (24). For simplicity we can choose the coefficients C_i so that terms proportional to $\partial x^0 \cdot \partial x^3$ never appear in the stress tensor. Thus, we have to choose $C_1 = 0$, which implies $C_2 = 1/\sqrt{k\lambda}$, $C_3 = 1/\sqrt{\mu k}$ and $C_4 = 0$. Then,

$$: T^2 := k\lambda : (\partial x^3)^2 : \quad (34)$$

$$: J^2 := \mu k : (\partial x^0)^2 : \quad (35)$$

Finally, putting together eqs. (33), (34) and (35) in the expression for $T(z)$, eq. (20)

$$T(z) = T_p(z) - \frac{k}{2(k-1)} (\partial X^-)^2 + \frac{\mu k}{2(k-1)} (\partial x^0)^2 - \frac{1}{2} (\partial x^3)^2$$

we observe that the parafermionic and bosonic contributions to the Sugawara stress tensor decouple:

$$T(z) = T_p(z) + \frac{\mu}{2} (\partial x^0)^2 - \frac{1}{2} (\partial x^3)^2$$

and the central charge of the full algebra is the sum of the central charge of the free boson fields, which add up to $c_x = 2$, and the parafermionic fields, which contribute $c_p = (2k+1)/(k-1)$. Thus, the full central charge is given by:

$$c = c_x + c_p = 2 + \frac{2k+1}{k-1} = 4 + \frac{3}{k-1}$$

as in eq. (19), which confirms that we actually have a representation of the original WZW model.

Once we have represented the current algebra in terms of bosonic and parafermionic fields, we can construct irreducible representations of the

$SL(2, R) \times R$ Lie algebra, that will serve as the base for the current algebra representations.

There exist two independent Casimir operators, one is linear: T , and the other is quadratic:

$$C^{(2)} = \Omega_{ab} J^a J^b = P_1^2 + P_2^2 - J^2 + \lambda T^2 = \frac{1}{2} (P^+ P^- + P^- P^+) - J^2 + \lambda T^2$$

We begin by defining the eigenstates of the Cartan subalgebra

$$J | j, t \rangle = -ij | j, t \rangle$$

$$T | j, t \rangle = -it | j, t \rangle$$

The action of P^\pm on these states can be evaluated using the hermiticity condition $(P^\pm)^\dagger = P^\mp$

$$P^\pm | j, t \rangle = \sqrt{C^{(2)} - \lambda t^2 + j(j \pm 1)} | j \pm 1, t \rangle \quad (36)$$

where $C^{(2)}$ is the eigenvalue of the quadratic Casimir. P^\pm act as raising and lowering operators, respectively.

We can distinguish various types of infinite dimensional representations according to the existence of lowest (lw) or highest (hw) weight states. The (hw) and (lw) representations are equivalent choices related by a discrete symmetry: $J \rightarrow -J$. They are characterized by the values of $C^{(2)}$, t and j . Defining $c^{(2)} \equiv C^{(2)} - \lambda t^2$, we see from (36) that in order to get (lw) or (hw) representations we must demand

$$c^{(2)} + j(j \pm 1) = 0 \quad (37)$$

for a particular value of $c^{(2)}$, that we will assume $\leq 1/4$, as will be clear below.

a) Lowest weight representations.

For these representations we have $P^+ | j, t \rangle = 0$. There are two values of j that satisfy this condition:

$$j_{\pm}^{(lw)} = \frac{-1 \pm \sqrt{1 - 4c^{(2)}}}{2}.$$

For $j \geq j_+^{(lw)}$ and $j \leq j_-^{(lw)}$ we have $\sqrt{c^{(2)} + j(j+1)} \in R$, avoiding in this way the zero norm eigenstates of the Casimir $C^{(2)}$. The (lw) representations are characterized by the values

$$j = j_-^{(lw)} - n \quad (38)$$

for any natural number n . This can be seen from the fact that acting with P^+ repetitively we can raise the index j until the condition $(P^+)^n |j, t\rangle = 0$ is reached. The spectrum of iJ is $j, j-1, j-2, \dots$.

b) Highest weight representations.

For these representations we need $P^- |j, t\rangle = 0$. In this case there are also two values of j satisfying this condition,

$$j_{\pm}^{(hw)} = \frac{1 \pm \sqrt{1 - 4c^{(2)}}}{2}.$$

For $j \geq j_+^{(hw)}$ and $j \leq j_-^{(hw)}$ we have $\sqrt{c^{(2)} + j(j-1)} \in R$, and the (hw) representations are characterized by the values

$$j = j_+^{(hw)} + n, \quad (39)$$

so that $(P^-)^n |j, t\rangle = 0$. The spectrum of iJ is $j, j+1, j+2, \dots$.

c) Other representations.

When the values of j do not meet the values given by (38) or (39) the representations are neither (lw) nor (hw). In these cases we can act with P^{\pm} freely, and the representations are not bounded above or below, but the square root in (36) becomes imaginary when j is in the interval $j_+^{(lw)} \leq j \leq j_-^{(lw)}$ or $j_+^{(hw)} \leq j \leq j_-^{(hw)}$.

When $c^{(2)} > 1/4$ the square root in (36) becomes complex, and the representations are neither (hw) nor (lw). The only representation which has both a lowest and a highest weight is the unit-like representation: $c^{(2)} = j = 0$.

It is useful to calculate the value of the zero mode of the Sugawara stress tensor (20),

$$L_0 = \frac{-1}{2(k-1)} [C^{(2)} - \lambda t^2 + 2j^2] + \frac{1}{2k\lambda} t^2$$

Once we have the representations of the Lie group we can use them to construct the current algebra representations acting with the negative modes of the currents. The vertex operators W_i that create the primary states must obey the following OPEs,

$$J_a(z)W_i(w) = T_{ij}^a \frac{W_j(w)}{(z-w)} + regular \quad (40)$$

where the coefficients T_{ij}^a are representation matrices for the Lie algebra. The dependence of the vertex operators on x^0 and x^3 can be made explicit in the following way

$$W(z) \sim e^{-ip_0 x^0 + ip_3 x^3} S(\psi_{\pm 1})$$

where S does not depend on the bosonic fields x^0 or x^3 . Applying the Cartan subalgebra J and T (for $\mu = +1$), yields

$$J(z) W(w) = -ip_0 \sqrt{k} \frac{W(w)}{(z-w)} + \text{regular}$$

$$T(z) W(w) = -ip_3 \sqrt{\lambda k} \frac{W(w)}{(z-w)} + \text{regular}$$

So that, in this parametrization, the identification

$$j = p_0 \sqrt{k}$$

$$t = p_3 \sqrt{\lambda k}$$

can be made. P^\pm actually act as raising and lowering operators,

$$P^\pm(z) : e^{-ip_0 x^0(w) + ip_3 x^3(w)} : \sim : e^{-i(p_0 \pm \frac{1}{\sqrt{k}})x^0(w) + ip_3 x^3(w)} : \psi_{\pm 1}(w) (z-w)^{\mp p_0 \frac{1}{\sqrt{k}}} + \dots$$

changing $j \rightarrow j \pm 1$. In order to satisfy the OPE (40), we must have

$$\psi_{\pm 1}(z) S(w) \sim (z-w)^{-1 \pm p_0 \frac{1}{\sqrt{k}}}$$

It is possible to adopt a free field realization of the current algebra (31), which represents the two parafermionic currents $\psi_{\pm 1}$ as, [21]-[22]

$$\psi_{\pm 1} = \frac{1}{2\sqrt{k}} \left[\pm \sqrt{2(k-1)} \partial_z x^1(z) - i\sqrt{2k} \partial_z x^2(z) \right] e^{\pm i\sqrt{\frac{1}{k}} x^2(z)}$$

where x^1 and x^2 are two free bosons

$$\langle x^i(z) x^j(w) \rangle = -\delta^{ij} \ln(z-w)$$

The parafermionic stress tensor, expressed in these bosonic fields is

$$T_p(z) = -\frac{1}{2} (\partial_z x^1)^2 - \frac{1}{2} (\partial_z x^2)^2 + \frac{1}{2\sqrt{k-1}} \partial_z^2 x^1$$

which is a Coulomb-gas representation with a background charge placed at infinity. The full stress tensor is

$$T(z) = -\frac{1}{2} \eta_{\mu\nu} \partial_z x^\mu \partial_z x^\nu + \frac{1}{2\sqrt{k-1}} \partial_z^2 x^1$$

with $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$.

Thus, the Sugawara stress tensor is written entirely in terms of four free bosons of Lorentzian signature with a background charge. This simplifies the treatment of physical states and the calculation of physical amplitudes, since the well known screening operator technics [23] may be used. The above results are essential in understanding string propagation in this cosmological spacetime background. Work in this direction is in progress.

6 Conclusions.

We have constructed a string theory on a homogeneous anisotropic four dimensional spacetime from a nonsemisimple Lie group. This spacetime is an expanding and contracting universe with constant scalar curvature.

By performing a duality transformation in an arbitrary direction in the transverse space, we found other expanding or contracting backgrounds with initial or final singularities. Since the duality transformations are valid only to lowest order in the α' expansion, string propagation in these dual spaces is only consistent to this lowest order.

A Sugawara construction was performed when the Lie algebra (1) possesses an invariant tensor L^{ab} and the currents obey the algebra (14) with Ω_{ab} given by equation (5). The Virasoro central charge is given by equation (19). Unlike the non-semisimple examples considered so far, the central charge does receive 1-loop corrections, thus it is non integer in general as observed in reference [24]. It is possible to factorize this construction into a standard (semisimple) Sugawara part and a nonsemisimple one (with integral central charge). In the semi-classical limit ($k \rightarrow \infty$), we recover $c = 4$, and it is possible to map the theory in terms of four free bosonic fields. In the general case, the conformal theory may be represented by two bosons and two parafermions, or equivalently, by four free bosons with a background charge placed at infinity. This may indicate a connection between the conformal models corresponding to the cosmological background and to another flat spacetime with a linear dilaton field in one of the spatial directions. The order K of the parafermion model depends on the level of the affine Kac-Moody algebra, as well as the background charge in the free boson representation.

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